ON HIGHER ANALOGS OF TOPOLOGICAL COMPLEXITY

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ABSTRACT. Farber introduced a notion of topological complexity $\mathrm{TC}(X)$ that is related to robotics. Here we introduce a series of numerical invariants $\mathrm{TC}_n(X), n = 2, 3, \dots$ such that $\mathrm{TC}_2(X) = \mathrm{TC}(X)$ and $\mathrm{TC}_n(X) \le \mathrm{TC}_{n+1}(X)$. For these higher complexities, we define their symmetric versions that can also be regarded as higher analogs of the symmetric topological complexity.

1. Introduction

In [F03] Farber introduced a notion of topological complexity $\mathrm{TC}(X)$ and related it to a problem of robot motion planning algorithm. Here we introduce a series of numerical invariants $\mathrm{TC}_n(X), n = 2, 3 \dots$ such that $\mathrm{TC}_2(X) = \mathrm{TC}(X)$ and $\mathrm{TC}_n(X) \le \mathrm{TC}_{n+1}(X)$. We learn some properties of TC_n and, in particular, compute $\mathrm{TC}_n(S^k)$. We also define symmetric analogs of higher complexities (=higher analogs of symmetric complexity) introduced in [F06, Section 31] and developed in [FG07, GL09].

Throughout the paper $\operatorname{cat} X$ denotes the Lusternik–Schnirelmann category of a space X, i.e. $\operatorname{cat} X$ is one less than the minimal of open and contractible sets in X that cover X. For example, X is contractible iff $\operatorname{cat} X = 0$.

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2. The Schwarz genus of a map

Given a map $f : X \to Y$ with X, Y path connected, a *fibrational substitute* of f is defined as a fibration $\widehat{f} : E \to Y$ such that there

exists a commutative diagram

$$\begin{tikzcd} X \arrow[r, "h"] \arrow[d, "f"'] & E \arrow[d, "\widehat{f}"] \\ Y \arrow[r, equal] & Y \end{tikzcd}$$

where h is a homotopy equivalence. The well-known result of Serre [S51] tells us that every map has a fibrational substitute, and it can be proved that any two fibrational substitutes of a map are fiber homotopy equivalent fibrations.

Given a map $f : X \to Y$, we say that a subset A of Y is a local f-section if there exists a map $s : A \to X$ (a local section) such that $fs = \mathrm{id}$.

The Schwarz genus of a fibration $p : E \to B$ is defined as a minimum number k such that there exists an open covering $U_1, \dots, U_k$ of B where each map U_i has a local p-section, [Sva66]. We define the Schwarz genus of a *map* f as the Schwarz genus of its fibrational substitute, and we denote it by $\mathfrak{genus}(f)$. This notion is well-defined since any two fibrational substitutes of a map are fiber homotopy equivalent.

2.1. **Proposition.** *For any diagram* $X \xrightarrow{f} Y \xrightarrow{g} Z$ *we have* $\mathfrak{genus}(gf) \geq \mathfrak{genus}(g)$.

Proof. This is clear if both f and g (and therefore gf) are fibrations. In the general case, replace f and g by fibrational substitutes. □

The following remark is useful for applications.

2.2. **Proposition.** *Let $p : E \to B$ be a fibration over a polyhedron B. Suppose that $B = X_1 \cup \dots \cup X_n$ where each X_i is an ENR and has a local p-section. Then $\mathfrak{genus}(f) \leq n$.*

Proof. We enlarge each X_i to an open subset of B over which there is a section of p. Take an ENR $X_i = X$ an embedding $X \subset B \subset \mathbb{R}^N$. Let $r : V \to X$ be a neighborhood retraction. Then there exists an open set U of V with $X \subset U \subset V$ such that the maps $U \subset V$ and $U \subset V \xrightarrow{r} X \subset V$ are homotopic, [D95, Chapter 4, especially 8.6, 8.7]. So, there is a homotopy $H : U \times I \to V$, $H(u, 0) = u, H(u, 1) \subset X$. Consider a section $s : X \to E$ and put $g : U \to E, g(u) = sH(u, 1)$. Now use the homotopy extension property to construct a homotopy $G : U \times I \to E$ with $pG = H$ and $G(u, 1) = g(u)$. Then $\sigma : U \to E$, $\sigma(u) = G(u, 0)$ is a section over U. □

3. Higher topological complexity

3.1. **Definition.** Let $J_n, n \in \mathbb{N}$ denote the wedge of n closed intervals $[0,1]_i, i = 1, \dots n$ where the zero points $0_i \in [0,1]_i$ are identified. Consider a path connected space X and set $T_n(X) := X^{J_n}$. There is an obvious map (fibration) $e_n : T_n(X) \to X^n, e_n(f) = (f(1_1), \dots, f(1_n))$ where 1_i is the unit in $[0,1]_i$, and we define $\mathrm{TC}_n(X)$ to be the Schwarz genus of e_n.

3.2. **Remarks.** 1. The above definition makes also sense for $\mathrm{TC}_1(X)$ (to be always equal to 1), but the notation that are started from $\mathrm{TC}_n, n > 1$ turns out to be more elegant.
2. It is easy to see that $\mathrm{TC}_n(X) \geq \mathrm{TC}_n(Y)$ if X dominates Y. So, TC_n is a homotopy invariant.
3. It is also worth noting that the fibration e_n can be described as follows: Take the diagonal map $d_n : X \to X^n$ and regard e_n as its fibrational substitute à la Serre. Hence, in fact, the higher topological complexity $\mathrm{TC}_n(X)$ is the Schwarz genus of the diagonal map $d_n : X \to X^n$. Note also that the (homotopy) fiber of e_n is $(\Omega X)^{n-1}$ where ΩX denotes the loop space of X.
4. The fibration e_n is homotopy equivalent to the following fibration e'_n. Define $S_n(X) \subset X^I \times X^n$ as

$$S_n(X) = \{(\alpha, x_1, \dots, x_n) \bigm| x_i \in \mathrm{Im}(\alpha : I \to X, i = 1, \dots, n)\}$$

and define $e'_n : S_n(X) \to X^n$ as $e'_n(\alpha, x_1, \dots, x_n) = (x_1, \dots, x_n)$. To prove that e'_n is a fibrational substitute of d_n, consider the homotopy equivalence $h : X \to S_n(X), h(x) = (\varepsilon_x, x, \dots, x)$ where ε_x is the constant path at x. Note that $e'_n h = d_n : X \to X^n$, and thus e'_n is the fibrational substitute of d_n.
5. The fibration e_n is homotopy equivalent to the fibration

$$e''_n : X^I \to X^n, e''_n(\alpha) = \left(\alpha(0), \alpha\left(\frac{1}{n-1}\right), \dots, \alpha\left(\frac{k}{n-1}\right), \dots, \alpha(1)\right)$$

where $\alpha : I \to X$. Indeed, consider the homotopy equivalence $h : X \to X^I, h(x) = \varepsilon_x$, and note that $e''_n h = d_n$.
6. It is easy to see (especially in view of the previous item) that $\mathrm{TC}_2(X)$ coincides with the topological complexity $\mathrm{TC}(X)$ introduced by Farber [F03].
7. Mark Grant pointed out to me that, similarly to $\mathrm{TC}_2(X)$, the invariant $\mathrm{TC}_n(X)$ is related to robotics. In detail, $\mathrm{TC}_2(X)$ is related to motion planning algorithm when a robot moves from a point to another

point, while $\mathrm{TC}_n(X)$ is related to motion planning problem whose input is not only an initial and final point but also an additional $n-2$ intermediate points.

3.3. **Proposition.** $\mathrm{TC}_n(X) \le \mathrm{TC}_{n+1}(X)$.

Proof. Let $d_k : X \to X^k$ denote the diagonal, $d_k(x) = (x, \dots, x)$. Note that $\mathrm{TC}_k(X)$ is the Schwarz genus of the map d_k. Define

$$\varphi : X^n \to X^{n+1}, \varphi(x_1, \dots, x_{n-1}, x_n) = (x_1, \dots, x_{n-1}, x_n, x_n).$$

Then $d_{n+1} = \varphi d_n$, and hence the Schwarz genus of d_{n+1} is greater than or equal to the Schwarz genus of d_n by Proposition 2.1. □

To compute TC_n, we can apply known methods of calculation of the Schwarz genus. For example, the Schwarz genus of a fibration over B does not exceed $1 + \operatorname{cat} B$. So,

$$\mathrm{TC}_n(X) \le 1 + \operatorname{cat}(X^n) \le n \operatorname{cat} X + 1. \tag{3.1}$$

Furthermore, we have the following claim, [Sva66, Theorem 4] (here, generally, $H^*(X; A_i)$ denotes cohomology with twisted coefficients).

3.4. **Proposition.** *Let $d_n : X \to X^n$ be the diagonal. If there exist $u_i \in H^*(X^n; A_i), i = 1, \dots, m$ so that $d_n^* u_i = 0$ and*

$$u_1 \smile \cdots \smile u_m \neq 0 \in H^*(X^n; A_1 \otimes \cdots \otimes A_m),$$

then $\mathrm{TC}_n(X) \ge m + 1$. □

3.5. **Proposition.** *If X is a connected finite CW-space that is not contractible, then* $\mathrm{TC}_n(X) \ge n$ *.*

Proof. If X is $(k-1)$-connected with $k > 1$ then $H^k(X; \mathbb{F}) \neq 0$ for some field $\mathbb{F}$. Take a non-zero $v \in H^k(X; \mathbb{F})$ and put $v_i = p_i^* v$ where $p_i : X^n \to X$ is the projection onto the ith factor. Then $u_i := v_i - v_n \in \operatorname{Ker} d_n^*$ for $i = 1, \dots, n-1$ and $u_1 \smile \cdots \smile u_{n-1} \neq 0$, and so $\mathrm{TC}_n(X) \ge n$ by Proposition 3.4.

Now, assume that X is not simply connected. Take the Berstein class $v \in H^1(X; I)$ where I is the augmentation ideal in the integral group ring of $\pi_1(X)$, see [B76, DR09]. Then argue as in the previous paragraph. □

4. An Example: $\mathrm{TC}_n(S^k)$

Farber [F03, Theorem 8] proved that $\mathrm{TC}(S^k) = 2$ for k odd and $\mathrm{TC}(S^k) = 3$ for k even. We extend this result (and method) and show that $\mathrm{TC}_n(S^k) = n$ for k odd and $\mathrm{TC}_n(S^k) = n+1$ for k even. Fix $n > 2$ and $k > 0$.

For k even, take a generator $u \in H^k(S^k) = \mathbb{Z}$ and denote by u_i its image in the copy S^k_i of S^k, $i = 1, \dots, n$. In the class $H^k((S^k)^n)$, consider the element

$$v = \left(\sum_{i=1}^{n-1} 1 \otimes \cdots \otimes 1 \otimes u_i \otimes 1 \otimes \cdots \otimes 1 \right) - 1 \otimes \cdots \otimes 1 \otimes (n-1)u_n.$$

Then $v^n = (1-n)n!(u_1 \otimes \cdots \otimes u_n)$ since k is even, and so $v^n \neq 0$. On the other hand, $d_n^* v = 0$. Thus, $\mathrm{TC}_n(S^k) = n+1$ by (3.1) and Proposition 3.4.

Now we prove that $\mathrm{TC}_n(S^k) = n$ for k odd. Consider a unit tangent vector field V on S^k, $V = \{V_x \bigm| x \in S^k\}$. Given $x, y \in S^k$ such that y is the antipode of x, denote by $[x, y]$ the path $[0, 1]$ determined by the geodesic semicircle joining x to y and such that the V_x is the direction of the semicircle at x.

Furthermore, if x and y are not antipodes, denote by $[x, y]$ the path $[0, 1]$ determined by the shortest geodesic from x to y.

Define an injective (non-continuous) function

$$\varphi : (S^k)^n \longrightarrow T_n(S^k),$$
$$\varphi(x_1, \dots, x_n) = \{[x_1, x_1], \dots, [x_1, x_n]\}.$$

For each $j = 0, \dots, n-1$ consider the submanifold (with boundary) U_j in $(S^k)^n$ such that each n-tuple $(x_1, \dots, x_n)$ in U_j has exactly j antipodes to x_1. Then $\varphi|_{U_j} : U_j \to T_n(S^k)$ is a continuous section of e_n, and $\bigcup_{i=0}^{n-1} U_i = (S^k)^n$. Furthermore, each $U_i, i = 0, \dots, n-1$ is an ENR, and so $\mathrm{TC}_n(S^k) \leq n$ by Proposition 2.2. Thus, $\mathrm{TC}_n(S^k) = n$ by Proposition 3.5.

5. Sequences $\{\mathrm{TC}_n(X)\}$

Of course, it is useful and interesting to compute invariants $\mathrm{TC}_n(X)$ for different spaces.

However, there is a general problem: to describe all possible (non-decreasing) sequences that can be realized as $\{\mathrm{TC}_n(X)\}_{n=1}^{\infty}$ with some fixed X.

As a first step, note that the inequality $\mathrm{TC}(X) \geq 1 + \operatorname{cat} X$ ([F08, Proposition 4.19]) together with (3.1) imply that

$$\mathrm{TC}_n(X) \leq n\,\mathrm{TC}_2(X) - n + 1. \tag{5.1}$$

So, any sequence $\{\mathrm{TC}_n(X)\}$ has linear growth.

Given $a \in \mathbb{N}$, we can also consider two functions

$$f_a(n) = \max_X \{\mathrm{TC}_n(X) \mid \mathrm{TC}(X) = a\}$$

and

$$g_a(n) = \min_X \{\mathrm{TC}_n(X) \mid \mathrm{TC}(X) = a\}.$$

So,

$$n \leq g_a(n) \leq f_a(n) \leq na - n + 1 \tag{5.2}$$

We can ask about the evaluation of the functions f_a and g_a. (This question was inspired by a discussion with M. Grant.)

Now we show that $g_3(n) < f_3(n)$ for $n > 2$.

We have $\mathrm{TC}(S^2) = 3 = \mathrm{TC}(T^2)$ (here T^2 is the 2-torus, the last equality can be found in [F03, Theorem 13]).

5.1. **Proposition.** $\mathrm{TC}_n(T^2) \geq 2n - 1$.

Proof. Let x, y be the canonical generators of $H^1(T^2)$. Put $x_i = p_i^* x$ where $p_i : (T^2)^n \to T^2$ is the projection on ith factor. Similarly, put $y_i = p^* y$. Then $d_n^*(x_2 - x_i) = 0 = d_n^*(y_2 - y_i)$ for $i = 2, \ldots, n$. On the other hand, the product

$$(x_2 - x_1) \smile \cdots \smile (x_n - x_1) \smile (y_2 - y_1) \smile \cdots \smile (y_n - y_1)$$

is non-zero. Indeed, it maps to $x_2 \smile \cdots \smile x_n \smile y_2 \smile \cdots \smile y_n \neq 0$ under the inclusion $(T^2)^{(n-1)} \to (T^2)^n$ on the last $n-1$ copies of T^2.

Now the claim follows from Proposition 3.4. $\square$

Thus, for $n > 2$ we have

$$g_3(n) \leq \mathrm{TC}_n(S^2) = n + 1 < 2n - 1 \leq \mathrm{TC}_n(T^2) \leq f_3(n).$$

So, we see that the sequence $\{\mathrm{TC}_n(X)\}$ contains more information on (the complexity of) a space X than just the number $\mathrm{TC}(X)$.

6. Symmetric topological complexity

Farber [F06, Section 31] considered a symmetric version $\mathrm{TC}^S(\mathrm{X})$ of the topological complexity. More detailed information about this invariant can be found in the papers Farber–Grant [FG07] and González–Landweber [GL09]. We define its higher analogs $\mathrm{TC}_n^S(X)$ as follows: Let $\Delta = \Delta_X^n \subset X^n$ be the discriminant,

$$\Delta = \{(x_1, \dots, x_n) \mid x_i = x_j \text{ for some pair } (i,j) \text{ with } i \neq j \}$$

The space $X^n \setminus \Delta$ consists of ordered configurations of n distinct points in X and is frequently denoted by $F(X,n)$. Let $v_n : Y \to F(X,n)$ be the restriction of the fibration e_n. Then the symmetric group Σ_n acts on Y by permuting paths and on $F(X,n)$ by permuting coordinates. These actions are free and the map v_n is equivariant. So, the map v_n yields a map (fibration) ev_n of the corresponding orbit spaces, and we define $\mathrm{TC}_n^S(X)$ as $\mathrm{TC}_n^S(X) = 1 + \mathfrak{genus}(ev_n)$. Note that, for the symmetric complexity we have $\mathrm{TC}^S(X) = \mathrm{TC}_2^S(X)$.

It is worth mentioning that in case $X = \mathbb{R}^2$ the space $F(X,n)/\Sigma_n$ is the classifying space for the n-braid group β_n. So, the symmetric topological complexity TC_n^S turns out to be related to the topological complexity of algorithms considered by Smale [Sm87] and Vassiliev [V88].

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